Direct CAPture (DCAP)

Notes on Formalism and Implementation

Olive S. Kirsebom

April 2020

1 Cross-section formula

According to Rolfs 1973, the "theoretical" cross section for E1 direct capture is given by,

$$\sigma_{\mathrm{theo}}(l_{\mathrm{i}}l_{\mathrm{f}}) \;=\; 0.0716\;\mu^{3/2}\left(\frac{Z_1}{M_1}-\frac{Z_2}{M_2}\right)^2 \frac{E_\gamma^3}{E_{\mathrm{i}}}^{3/2}\;\omega\,(l_{\mathrm{i}}010|l_{\mathrm{f}}0)^2\;R^2_{l_{\mathrm{i}}1l_{\mathrm{f}}}\,\mu\mathrm{b}\;, \quad (1)$$

where,

- Z_1, M_1 are the charge and mass numbers of the projectile
- Z_2, M_2 are the charge and mass numbers of the target
- $\mu = \frac{M_1 M_2}{M_1+M_2}$ is the reduced mass number
- E_γ is the γ-ray energy in MeV
- E_{i} is the energy of relative motion of the projectile and target in the c.m. frame in the initial state in MeV*
- $\omega = \frac{(2J_f+1)(2l_{\mathrm{i}}+1)}{(2J_1+1)(2J_2+1)(2l_{\mathrm{f}}+1)}$ is the spin statistical weight factor
- $(l_{\mathrm{i}}010|l_{\mathrm{f}}0)$ is the Clebsch-Gordan coefficient
- J_1, J_2, J_f are the spins of the projectile, target, and final state
- l_{i}, l_{f} are the orbital angular momenta in the initial and final state
- $R_{l_{\mathrm{i}}1l_{\mathrm{f}}}$ is the radial integral in units of fm$^{3/2}$

*Note that Rolfs is a little unclear on which energy should be used here. It is either the energy of the relative motion in the c.m. frame or the energy of the projectile in the c.m. frame. These energies differ by a factor of $M_2/(M_1+M_2)$.

The “experimental“ cross section is then found as,

$$\sigma_{\rm exp} \;=\; \sum_{l_{\rm f}} \left(T_1 T_{z,1} T_2 T_{z,2} | T_{\rm f} T_{z,{\rm f}}\right)^2 S(l_{\rm f}) \sum_{l_{\rm i}} \sigma_{\rm theo}(l_{\rm i} l_{\rm f}) \;, \tag{2}$$

where,

- $T_{z,1} = (M_1 - 2Z_1)/2$, $T_1 = |T_{z,1}|$ are the isospin quantum numbers of the projectile
- $T_{z,2} = (M_2 - 2Z_2)/2$, $T_2 = |T_{z,2}|$ are the isospin quantum numbers of the target
- $T_{z,{\rm f}} = T_{z,1} + T_{z,2}$, $T_{\rm f}$ are the isospin quantum numbers of the final state
- $(T_1 T_{z,1} T_2 T_{z,2} | T_{\rm f} T_{z,{\rm f}})$ is the isospin Clebsch-Gordan coefficient
- $S(l_{\rm f})$ is the spectroscopic factor

2 Radial integral

The radial integral is given by,

$$R_{l_{\rm i} 1 l_{\rm f}} \;=\; \int_0^\infty u_{\rm c}(r)\; \mathcal{O}_{\rm E1}(r)\; u_{\rm b}(r)\; r^2 \;{\rm d}r \;, \tag{3}$$

where,

- r is the projectile-target separation in fm
- $\mathcal{O}_{\rm E1}(r) = \left[(\rho^2 - 2)\sin\rho + 2\rho\cos\rho\right] 3r/\rho^3$ is the E1 multipole operator and $\rho = k_\gamma r$ where $k_\gamma = \frac{E_\gamma}{\hbar c}$
- $u_{\rm c}(r)$ is the initial-state (continuum) radial wave function
- $u_{\rm b}(r)$ is the final-state (bound) radial wave function

Note that $u_{\rm c}(r)$ has units of fm^{-1}, $u_{\rm b}(r)$ has units of fm$^{-3/2}$, and $\mathcal{O}_{\rm E1}(r)$ has units of fm, so the integrand has units of fm$^{1/2}$ and the integral has units of fm$^{3/2}$.

3 Continuum radial wave function, $u_{\mathbf{c}}(r)$

The continuum radial wave function has units of fm$^{-1/2}$ and is given by,

$$u_{\rm c}(r) \;=\; \exp\left[i(\sigma_{l_{\rm i}} - \sigma_0 + \delta_{l_{\rm i}}\right] \frac{u_{l_{\rm i}}(r)}{r} \;, \tag{4}$$

where,

$$\frac{u_{l_{\rm i}}(r)}{r} \;=\; \begin{cases} A\; j_{l_{\rm f}}(K_{\rm i} r)\;, & r \le R_0 \\ \left[F_{l_{\rm i}}(k_{\rm i} r)\cos\delta_{l_{\rm i}} + G_{l_{\rm i}}(k_{\rm i} r)\sin\delta_{l_{\rm i}}\right]/r\;, & r > R_0 \end{cases} \tag{5}$$

and,

- $R_0 = r_0(M_1^{1/3} + M_2^{1/3})$ is the nuclear radius, where the value $r_0 = 1.36$ fm is used
- j_{l_f} is the regular spherical Bessel function of order l_f
- F_{l_i}, G_{l_i} are the spherical Coulomb functions
- $K_\mathrm{i} = [2m(E_\mathrm{i} + V_0)]^{1/2}/\hbar$, where $m = \mu m_\mathrm{u}$ is the reduced mass, $E_\mathrm{i} > 0$ is the energy of relative motion of the projectile and target in the c.m. frame in the initial (continuum) state, and $V_0 > 0$ is the depth of the square-well potential
- $k_\mathrm{i} = (2mE_\mathrm{i})^{1/2}/\hbar$
- $\sigma_{l_\mathrm{i}} - \sigma_0$ is the usual Coulomb phase difference
- δ_{l_i} is the nuclear phase shift*
- A is a normalisation constant which is adjusted so that the wave function is continuous at $r = R_0$

*Note that Rolfs assumed the nuclear phase shifts to be given by the hard-sphere phase shifts at the nuclear radius R_0.

4 Bound radial wave function, $u_\mathbf{b}(r)$

The bound radial wave function has units of fm$^{-3/2}$ and is given by,

$$u_\mathrm{b}(r) = \frac{u_{l_\mathrm{f}}(r)}{r} = A \begin{cases} j_{l_\mathrm{f}}(K_\mathrm{f} r) \,, & r \le R_0 \\ B\, W_{\eta, l_\mathrm{f}}(\kappa_\mathrm{f} r)/r \,, & r > R_0 \end{cases} \tag{6}$$

where,

- j_{l_f} is the regular spherical Bessel function of order l_f
- W_{η, l_f} is the Whittaker function
- $K_\mathrm{f} = [2m(E_\mathrm{f} + V_0)]^{1/2}/\hbar$, where $m = \mu m_\mathrm{u}$ is the reduced mass, $E_\mathrm{f} < 0$ is the energy of relative motion of the projectile and target in the c.m. frame in the final (bound) state, and $V_0 > 0$ is the depth of the square-well potential
- $\kappa_\mathrm{f} = (-2mE_\mathrm{f})^{1/2}/\hbar$
- A, B are normalisation constants which are adjusted so that the wave function is continuous at $r = R_0$ and $\int_0^\infty u^*_{l_\mathrm{f}}(k_\mathrm{f} r) u_{l_\mathrm{f}}(k_\mathrm{f} r)\, \mathrm{d}r = 1$

The depth of the square-well potential, V_0, is adjusted to reproduce the binding energy of the final state, i.e., so that the logarithmic derivatives of the interior and exterior wave functions match at the channel radius,

$$\left[\frac{1}{j_{l_\mathrm{f}}(K_\mathrm{f}r)}\frac{\mathrm{d}}{\mathrm{d}r}j_{l_\mathrm{f}}(K_\mathrm{f}r)\right]_{r=R_0} = \left[\frac{r}{W_{\eta,l_\mathrm{f}}(\kappa_\mathrm{f}r)}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}W_{\eta,l_\mathrm{f}}(\kappa_\mathrm{f}r) - \frac{1}{r^2}W_{\eta,l_\mathrm{f}}(\kappa_\mathrm{f}r)\right)\right]_{r=R_0} \tag{7}$$